[12] Wieghart, K., "On the Blowing of Warm Air for De-Icing Devices," FB Nr 1900 (1944)—*Zentrale für wissenschaftliches Berichtswesen*, Nov. 1946, AAF Translation F-TS 919 RE.

[13] Seban, R. A., "Heat Transfer and Effectiveness for a Turbulent Boundary Layer With Tangential Fluid Injection," *Journal of Heat Transfer*, Nov. 1960, pp. 303–312.

[14] Hartnett, J. P., Birkebak, R. C., and Eckert, E. R. G., "Velocity Distributions, Temperature Distributions, Effectiveness and Heat Transfer for Air Injected Through a Tangential Slot Into a Turbulent Boundary Layer," Paper 60-HT-31, Aug. 15–17, 1960, presented at the ASME-AIChE Heat Transfer Conference, Buffalo, N. Y.

[15] Chin, J. H., Skirvin, S. C., Hayes, L. E., and Silver, A. H., "Adiabatic Wall Temperature Downstream of a Single, Tangential Injection Slot," Paper 58-A-107, present at the ASME Annual Meeting, New York, Nov. 30–Dec. 5, 1958.

[16] Papell, S. S. and Trout, A. M., "Experimental Investigation of Air Film Cooling Applied to an Adiabatic Wall by Means of an Axially Discharging Slot," TN D-9, Aug. 1959, NASA.

SEPTEMBER 1972 AIAA JOURNAL VOL. 10, NO. 9

Accurate Numerical Methods for Boundary-Layer Flows. II: Two-Dimensional Turbulent Flows

HERBERT B. KELLER*
California Institute of Technology, Pasadena, Calif.
AND
TUNCER CEBECI†
Douglas Aircraft Company, Long Beach, Calif.

A very simple and accurate numerical scheme which is applicable to quite general boundary-layer-flow problems has been devised. It has been tested extensively on laminar flows, turbulent flows (using eddy-diffusivity expressions), wake flows, and many other such flow problems. The procedure is much faster, easier to program and more flexible than most (if not all) other numerical methods which have been employed on such problems. This scheme also enables one to compute extremely close to the point of separation with no special precautions. The inclusion of chemically reacting species should offer no difficulties. The efficiency of the scheme clearly makes it applicable to three-dimensional steady, unsteady two-dimensional, and perhaps even unsteady three-dimensional, boundary-layer flows.

Nomenclature

A = damping length, see Eq. (6)
c_f = local skin-friction coefficient, $2\tau_w/\rho u_e^2$
f = dimensionless stream function
H = shape factor, δ^*/θ
p = pressure
p^+ = pressure-gradient parameter
R_x = local Reynolds number, $u_e x/\nu$
R_θ = Reynolds number based on momentum thickness, $u_e\theta/\nu$
u_τ = friction velocity, $(\tau_w/\rho)^{1/2}$
$\bar{u}, \bar{v}$ = x and y components of velocity
u_e = edge velocity
v_w^+ = dimensionless velocity ratio, $\bar{v}_w/u_\tau$
x, y = rectangular coordinates
β = velocity-gradient parameter
δ = boundary-layer thickness
δ^* = displacement thickness, $\int_0^\infty (1 - \bar{u}/u_e)\,dy$
ε = eddy viscosity
η = transformed y-coordinate
θ = momentum thickness, $\int_0^\infty \bar{u}/u_e\,(1 - \bar{u}/u_e)dy$
μ = dynamic viscosity
ν = kinematic viscosity
ξ = transformed x-coordinate
ρ = mass density
$-\rho\langle u'v'\rangle$ = Reynolds shear stress ‡
τ = shear stress, wherever applicable
ψ = stream function, wherever applicable

Subscripts

e = outer edge of boundary layer
i = inner region
o = outer region
w = wall

Presented as Paper 71-164 at the AIAA 9th Aerospace Sciences Meeting, New York, January 25–27, 1971; submitted May 6, 1971; revision received March 10, 1972.

Index category: Boundary Layers and Convective Heat Transfer—Turbulent.

* Professor of Applied Mathematics, Firestone Laboratories.

† Senior Engineer/Scientist, Aerodynamics Research Group. Member AIAA.

‡ Primes denote differentiation with respect to η.

Introduction

THERE is no shortage of numerical methods for solving, or more properly for approximating the solutions of various laminar and turbulent boundary-layer equations. The recent "Turbulent Olympics" volumes[1] give an excellent account of the current state-of-the-art for turbulent boundary-layer computations. The most crucial aspect of problems regarding turbulent flows is the basic formulation to be employed for the time-averaged fluctuating quantities that appear in the governing conservation equations. Since there is no generally accepted theory of turbulence, the field is essentially in a state of empiricism and many different theories or formulations yield results which agree well with a variety of experiments. On the other hand, each of these "theories" (i.e., eddy-viscosity, mixing length, turbulent energy, etc., Ref. 1) contain at least several parameters (if not functions) whose values are selected to best fit the results of some experiments. Thus, many of these theories resemble a sophisticated form of curve fitting. However, since they are of interest in many important problems in engineering, efficient and accurate numerical methods for their evaluation are of great importance. In addition, sufficiently flexible schemes enable three dimensional or time dependent problems to be treated and modifications to the empirical theories can easily be tested. This paper presents an extremely simple, accurate and efficient method that has been devised for these purposes.

The new method is much faster, easier to program and more flexible than most (if not all) other numerical methods which

have been employed on such problems. The scheme enables one to compute extremely close to the point of boundary-layer separation with no special precautions. The inclusion of energy or species equations offers no difficulties. The efficiency of the scheme clearly makes it applicable to three-dimensional, boundary-layer flows. Many of these applications will be reported elsewhere. The application of the method to incompressible laminar boundary-layers has been reported.[2] It has also been used[3] to solve the inverse problem of determining the pressure gradient required to produce a given wall shear (in laminar flows). Here the method is formulated in great detail for two-dimensional incompressible steady turbulent boundary-layers. Using the eddy-viscosity formulation[4] it is then applied to compute the turbulent flow over a flat plate and the results are compared with those obtained by the numerical method of Ref. 4.

The present method is based on a new accurate difference method that has been devised for parabolic partial differential equation.[5] One of the basic ideas of the procedure is to write the governing partial differential equations in the form of a first-order system. Thus, derivatives of some quantities with respect to the boundary-layer or "normal" variable must be introduced as new unknown functions. Derivatives with respect to all other, streamwise, variables occur only to first order as a consequence of the boundary-layer approximations. With the resulting first-order system and on an arbitrary rectangular net, we use simple centered difference quotients and averages at the midpoints of net rectangles or net segments, as required, to get $0(h^2)$ accurate finite difference equations.

This scheme is unconditionally stable but, of course, the equations are highly implicit and nonlinear. Newton's method is employed to solve them. In order to do this with an efficient and stable computational scheme, a block tridiagonal factorization technique is employed.

Finally, we show how Richardson extrapolation can be employed to improve the accuracy of the numerical solution. This can be done independently in each variable or jointly in all variables reducing the error by two orders of magnitude with respect to the mesh spacing in the relevant variables [for example for $0(\Delta x^2 + \Delta y^2)$ to $0(\Delta x^2 + \Delta y^4)$ or to $0(\Delta x^4 + \Delta y^4)$]. The use of these extrapolation techniques can reduce the computing effort dramatically.

Boundary-Layer Equations

If the normal stress terms are neglected, the usual boundary-layer equations for continuity and momentum of two-dimensional, steady, incompressible turbulent flows can be written as

$$\partial\bar{u}/\partial x + \partial\bar{v}/\partial y = 0 \tag{1}$$

$$\bar{u}\,\partial\bar{u}/\partial x + \bar{v}\,\partial\bar{u}/\partial y = u_e\,du_e/dx + \nu\,\partial^2\bar{u}/\partial y^2 - (\partial/\partial y)\langle u'v'\rangle \tag{2}$$

The boundary conditions are

$$\bar{u}(x,0) = 0,\ \bar{v}(x,0) = \bar{v}_w,\ \lim_{y\to\infty}\bar{u}(x,y) = u_e(x) \tag{3}$$

In order to solve the system given by Eqs. (1–3) it is necessary to relate the Reynolds shear-stress term, $-\rho\langle u'v'\rangle$, to mean velocity distributions. Here the eddy-viscosity formulation (5) is used to define

$$-\rho\langle u'v'\rangle = \rho\varepsilon\,\partial\bar{u}/\partial y \tag{4}$$

According to this formulation the turbulent boundary-layer is regarded as a composite layer characterized by inner and outer regions, and the eddy viscosity ε is represented by Prandtl's mixing-length together with modifications[6,7] and is written as

$$\varepsilon_i = (0.4y)^2[1 - \exp(-y/A)]^2\,|\partial\bar{u}/\partial y| \tag{5}$$

where A is a damping constant given by

$$A = (26\nu/u_\tau)\{-(p^+/v_w^+)[\exp(11.8v_w^+) - 1] + \exp(11.8v_w^+)\}^{-1/2} \tag{6}$$

with

$$p^+ \equiv -(dp/dx)\,\nu/\rho u_\tau^3,\quad v_w^+ \equiv v_w/u_\tau,\quad u_\tau \equiv (\tau_w/\rho)^{1/2}$$

In the outer region we use a constant eddy-viscosity expression with Klebanoff's intermittency factor,[8] to get

$$\varepsilon_0 = 0.0168\left|\int_0^\infty (u_e - \bar{u})dy\right|\left\{\{1 + 5.5(y/\delta)^6\}^{-1}\right. \tag{7}$$

The inner and outer eddy-viscosity expressions are matched by the obvious requirement of continuity. For more discussion and explicit choices of ε see Refs. 4 and 9.

It is convenient to introduce new independent variables to eliminate the singularity at $x = 0$ and to stretch the boundary-layer thickness. This is done by the Levy-Lees transformation[10] to get new variables (ξ, η) from

$$d\xi = \rho\mu u_e dx,\quad d\eta = \rho u_e(2\xi)^{-1/2}dy \tag{8}$$

Further, if $\psi(x, y)$ *is a stream function such that*

$$\partial\psi/\partial y = \bar{u},\quad \partial\psi/\partial x = -\bar{v} \tag{9}$$

then the continuity equation is satisfied. Introducing a dimensionless stream function $f(\xi,\eta)$ by

$$\psi(x,y) \equiv (2\xi)^{1/2}f(\xi,\eta) \tag{10}$$

and using the new variables of Eq. (8) the momentum equation becomes

$$[bf'']' + ff'' + \beta[1 - (f')^2] = 2\xi(f'\,\partial f'/\partial\xi - f''\,\partial f/\partial\xi) \tag{11}$$

Here the primes denote differentiation with respect to η. The boundary conditions (3) now become

$$f(\xi,0) = f_w,\quad f'(\xi,0) = 0,\quad \lim_{\eta\to\infty} f'(\xi,\eta) = 1 \tag{12}$$

In Eq. (11) we have introduced the coefficients

$$b = b(\xi,\eta,\beta,f',f'') \equiv 1 + v^{-1}\varepsilon(\xi,\eta,\beta,f',f'') \tag{13}$$

$$\beta = \beta(\xi) \equiv (2\xi/u_e)(du_e/d\xi)$$

Specific formulae for b are presented in Ref. 4. It should be observed that the standard laminar boundary-layer equations are obtained if $b \equiv 1$ in Eq. (11).

Numerical Formulation

A crucial step in the numerical procedure is to reformulate the problem in terms of a first-order system of partial differential equations. For this purpose we introduce new dependent variables $u(\xi,\eta)$ and $v(\xi,\eta)$ so that Eq. (11) can be written as§:

$$f' = u \tag{14a}$$

$$u' = v \tag{14b}$$

$$bv' = -(b' + f)v + \beta(u^2 - 1) + 2\xi(u\,\partial u/\partial\xi - v\,\partial f/\partial\xi) \tag{14c}$$

The boundary conditions (12) are simply

$$f(\xi,0) = f_w(\xi),\quad u(\xi,0) = 0,\quad \lim_{\eta\to\infty} u(\xi,\eta) = 1 \tag{15}$$

As is well known, the continuity and momentum equations in boundary-layer theory behave as a parabolic system. Indeed, in the laminar case, they can be reduced, by means of the von Mises transformation (11) to a scalar diffusion equation. Thus, we employ a numerical scheme which has been rather thoroughly analyzed for such problems[5] and which has numerous features that make it attractive in the present case. In particular, it has high-order accuracy with nonuniform net spacing, it is easily programed and is quite efficient.

Let the net points be given by

$$\xi_o = 0,\quad \xi_n = \xi_{n-1} + k_n,\quad n = 1,2,\ldots,N$$

$$\eta_0 = 0,\quad \eta_j = \eta_{j-1} + h_j,\quad j = 1,2,\ldots,J;\eta_J \equiv \eta_\infty \tag{16}$$

The net spacings, k_n and h_j, are completely arbitrary and indeed may have large variations in practical calculations. This is especially important in turbulent boundary-layer calculations which are characterized by large boundary-layer thicknesses. To get accuracy near the wall, small net spacing is required while large spacing can be used away from the wall.

§ It would have been preferable to retain the term $(bv)'$ in Eq. (14c). However by some lapse we have used it in the form $bu' + b'v$ in the actual calculations and so that is how we describe it here.

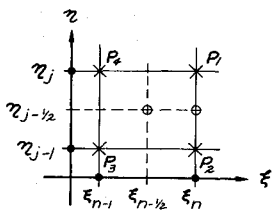

Fig. 1 Net rectangle for difference approximations.

The quantities (f, u, v) will be approximated at points (ξ_n, η_j) of of the net by net functions denoted by (f_j, u_j, v_j). We also employ the notation, for points and quantities midway between net points and for any net function g_j^n

$$\xi_{n-1/2} \equiv \tfrac{1}{2}(\xi_n + \xi_{n-1}), \quad \eta_{j-1/2} \equiv \tfrac{1}{2}(\eta_j + \eta_{j-1}) \tag{17a}$$

$$g_j^{n-1/2} \equiv \tfrac{1}{2}(g_j^n + g_j^{n-1}), \quad g_{j-1/2}^n \equiv \tfrac{1}{2}(g_j^n + g_{j-1}^n) \tag{17b}$$

The difference equations which are to approximate Eq. (14) are now easily formulated by considering one mesh rectangle as in Fig. 1. We simply approximate (Eqs. 14a, 14b) using centered difference quotients and average about the midpoint $(\xi_n, \eta_{j-1/2})$ of the segment P_1P_2. Similarly Eq. (14c) is approximated by centering about the midpoint $(\xi_{n-1/2}, \eta_{j-1/2})$ of the rectangle $P_1P_2P_3P_4$.

The only ambiguity or choice in the above indicated approximations concerns the nonlinear terms. We may take averages of products, as in $fu \sim fu_{j-1/2}^{n-1/2}$, or products of averages, as in $fu \sim f_{j-1/2}^{n-1/2} u_{j-1/2}^{n-1/2}$. Although the former alternative is frequently simpler, in the computations reported here we have used the latter alternative. Thus, our difference approximations to Eq. (14) are

$$f_j^n - f_{j-1}^n / h_j = \tfrac{1}{2}(u_j^n + u_{j-1}^n) \equiv u_{j-1/2}^n \tag{18a}$$

$$u_j^n - u_{j-1}^n / h_j = \tfrac{1}{2}(v_j^n + v_{j-1}^n) \equiv v_{j-1/2}^n \tag{18b}$$

$$b_{j-1/2}^n \left(\frac{v_j^n - v_{j-1}^n}{h_j}\right) = -\left(\frac{b_j^n - b_{j-1}^n}{h_j} + f_{j-1/2}^n\right) v_{j-1/2}^n - \beta_n(1 - (u_{j-1/2}^n)^2) + \alpha_n\{(u_{j-1/2}^n)^2 - f_{j-1/2}^n v_{j-1/2}^n - v_{j-1/2}^{n-1} f_{j-1/2}^n + v_{j-1/2}^n f_{j-1/2}^{n-1}\} + T_{j-1/2}^{n-1} \tag{18c}$$

where $\alpha_n = (2\xi_{n-1/2}/\Delta\xi_n)$ and

$$T_{j-1/2}^{n-1} = -\left\{b_{j-1/2}^{n-1}\left(\frac{v_j^{n-1} - v_{j-1}^{n-1}}{h_j}\right) + \left\{\frac{b_j^{n-1} - b_{j-1}^{n-1}}{h_j} + f_{j-1/2}^{n-1}\right\} v_{j-1/2}^{n-1} + \beta_{n-1}\{1 - (u_{j-1/2}^{n-1})^2\}\right\} + \alpha_n\{f_{j-1/2}^{n-1} v_{j-1/2}^{n-1} (u_{j-1/2}^{n-1})^2\}$$

Equations (18) are imposed for $j = 1, 2, \ldots, J$ and η_j is to be sufficiently large so that it is beyond the edge of the boundary layer. (With no essential difficulty, η_j may be increased as the calculations proceed downstream from $\xi_n = 0$. We discuss this point later. The boundary conditions (15) yield, at $\xi = \xi_n$

$$f_0^n = f_w(\xi_n), \quad u_0^n = 0, \quad u_J^n = 1 \tag{19}$$

Solution of the Difference Equations

When $(f_j^{n-1}, u_j^{n-1}, v_j^{n-1})$ are known for $0 \leqq j \leqq J$, then Eqs. (18) and (19) are a system of $3J + 3$ nonlinear equations for the determination of the $3J + 3$ unknowns (f_j^n, u_j^n, v_j^n), $j = 0, 1, \ldots, J$. This nonlinear system is solved by means of Newton's method. (It is perhaps tempting to linearize the differential equations first and then to solve these linearized equations by difference methods. But that procedure, sometimes called "quasilinearization," does not necessarily yield the same accuracy or efficiency as the procedure we follow.) For simplicity of notation write the unknowns at $\xi = \xi_n$ as $(f_j^n, u_j^n, v_j^n \equiv f_j, u_j, v_j)$. Then the system (18) can be written as

$$(f_j - f_{j-1}) - (h_j/2)(u_j + u_{j-1}) = 0 \tag{20a}$$

$$(u_j - u_{j-1}) - (h_j/2)(v_j + v_{j-1}) = 0 \tag{20b}$$

$$b_{j-1/2}(v_j - v_{j-1}) + (b_j - b_{j-1}) v_{j-1/2} + h_j f_{j-1/2} v_{j-1/2}(1 + \alpha_n) - \alpha_n h_j\{f_{j-1/2}^{n-1} v_{j-1/2} - v_{j-1/2}^{n-1} f_{j-1/2}\} - h_j(u_{j-1/2})^2(\beta + \alpha_n) + \beta h_j = T_{j-1/2}^{n-1} h_j \tag{20c}$$

Note that $T_{j-1/2}^{n-1}$ involves only known quantities if the solution is known on $\xi = \xi_{n-1}$. To solve Eqs. (20) and (19) by Newton's method introduce the iterates $\{f_j^{(i)}, u_j^{(i)}, v_j^{(i)}\}$, $i = 0, 1, 2, \ldots$, with initial values

$$f_0^{(0)} = f_w(\xi_n), \quad u_0^{(0)} = 0, \quad v_0^{(0)} = v_0^{n-1}$$

$$f_j^{(0)} \equiv f_j^{n-1}, \quad u_j^{(0)} \equiv u_j^{n-1}, \quad v_j^{(0)} \equiv v_j^{n-1}, \quad 1 \leqq j \leqq J - 1$$

$$f_J^{(0)} = f_J^{n-1}, \quad u_J^{(0)} = 1, \quad v_J^{(0)} = v_J^{n-1} \tag{21}$$

For the higher order iterates set

$$f_j^{(i+1)} = f_j^{(i)} + \delta f_j^{(i)}, \quad u_j^{(i+1)} = u_j^{(i)} + \delta u_j^{(i)}, \quad v_j^{(i+1)} = v_j^{(i)} + \delta v_j^{(i)} \tag{22}$$

Now insert these expressions in place of $\{f_j, u_j, v_j\}$ in Eq. (20) and drop the terms that are quadratic in $\{f_j^{(i)}, u_j^{(i)}, v_j^{(i)}\}$. This procedure yields the following linear system:

$$[\delta f_j^{(i)} - \delta f_{j-1}^{(i)}] - (h_j/2)[\delta u_j^{(i)} + \delta u_{j-1}^{(i)}] = r_{j-1/2}^{(i)} \tag{23a}$$

$$[\delta u_j^{(i)} - \delta u_{j-1}^{(i)}] - (h_j/2)[\delta v_j^{(i)} + \delta v_{j-1}^{(i)}] = s_{j-1/2}^{(i)} \tag{23b}$$

$$\delta v_j^{(i)}\{b_{j-1/2}^{(i)} + \tfrac{1}{2}[b_j^{(i)} - b_{j-1}^{(i)}] + (h_j/2)(1 + \alpha_n) f_{j-1/2}^{(i)} - \alpha_n(h_j/2) f_{j-1/2}^{n-1}\} + \delta f^{(i)}\{(h_j/2)(1 + \alpha_n) v_{j-1/2}^{(i)} + \alpha_n(h_j/2) v_{j-1/2}^{n-1}\} - h_j u_{j-1/2}^{(i)}(\beta + \alpha_n)\delta u_j^{(i)} = \delta v_{j-1}^{(i)}\{b_{j-1/2}^{(i)} - [\tfrac{1}{2}[b_j^{(i)} - b_{j-1}^{(i)}] + (h_j/2)(1 + \alpha_n) f_{j-1/2}^{(i)} - \alpha_n(h_j/2) f_{j-1/2}^{n-1}]\} - \delta f_{j-1}^{(i)}\{(h_j/2)(1 + \alpha_n) v_{j-1/2}^{(i)} + \alpha_n(h_j/2) v_{j-1/2}^{n-1}\} + h_j u_{j-1/2}^{(i)}(\beta + \alpha_n)\delta u_{j-1}^{(i)} + t_{j-1/2}^{(i)} \tag{23c}$$

for $j = 1, 2, \ldots, J$. Here we have introduced

$$r_{j-1/2}^{(i)} = f_{j-1}^{(i)} - f_j^{(i)} + u_{j-1/2}^{(i)} h_j \tag{24a}$$

$$s_{j-1/2}^{(i)} = u_{j-1}^{(i)} - u_j^{(i)} + v_{j-1/2}^{(i)} h_j \tag{24b}$$

$$t_{j-1/2}^{(i)} = v_{j-1}^{(i)} + v_j^{(i)} - \frac{h_j}{b_{j-1/2}^{(i)}}\left[\beta - T_{j-1/2}^{n-1} + \frac{b_j^{(i)} - b_{j-1}^{(i)}}{h_j} v_{j-1/2}^{(i)} + f_{j-1/2}^{(i)} v_{j-1/2}^{(i)}(1 + \alpha_n) - \alpha_n\{f_{j-1/2}^{n-1} v_{j-1/2}^{(i)} - v_{j-1/2}^{n-1} f_{j-1/2}^{(i)}\} - [u_{j-1/2}^{(i)}]^2(\beta + \alpha_n)\right] \tag{24c}$$

Recalling Eq. (3) and that we are currently solving at $\xi = \xi_n$, the notation $b_j^{(i)}$ represents

$$b_j^{(i)} \equiv b[\xi_n, \eta_j, \beta, u_j^{(i)}, v_j^{(i)}] \tag{24d}$$

The only departure from the strict application of Newton's method is in treating the terms $b_{j-1/2}(v_j - v_{j-1})$. We have in fact neglected terms of the form $[(\partial b_j/\partial u)\delta u]v$ and $[(\partial b_j/\partial v)\delta v]v$ that should have included on the left-hand side of Eq. (23) as well as similar terms on the right-hand side. It is observed in the calculations that second-order convergence of the iterations is not noticeably affected by this lapse. Further, it should be observed that $b(\xi, \eta, \beta, u, v)$ in general has only piecewise continuous derivatives since it has different representations in the inner and outer regions of the boundary layer. Finally, note that since b is also a function of the boundary-layer thickness, it is a functional of u (involving an integral over $0 \leqq \eta \leqq \infty$). Hence, the exact treatment of this dependence would couple all values $\delta u_j^{(i)}$, $j = 0, 1, \ldots, J$, in each equation of the form (23) and complicate the solution of the resulting linear system in an unacceptable manner.

To complete the system (23) note that the boundary conditions (19) can be satisfied exactly with no iterations. Then in Eq. (21) the initial values of $f_0^{(0)}$, $u_0^{(0)}$ and and $u_j^{(0)}$ are altered accordingly (i.e., set $f_0^{(0)} = f_w(\xi_n)$, $u_0^{(0)} =$ and $u_j^{(0)} = 1$) and to maintain these correct values in all the iterates we take

$$\delta f_0^{(i)} = 0, \quad \delta u_0^{(i)} = 0, \quad \delta u_J^{(i)} = 0, \quad i = 0, 1, \ldots \tag{25}$$

The linear system (23), (25) can be solved in an extremely efficient manner since it has a block tri-diagonal structure. This is not obvious, so to clarify the solution procedure we write the system in matrix-vector form. There are many ways in which this can be done. They are all equivalent and merely amount to different permutations of the equations or of the unknowns or both. Further, the boundary conditions (25) could be employed

to eliminate three unknowns, and thus, slightly reduce the order of the system. The latter procedure has been employed in most of our calculations. But we proceed in a slightly different manner here and present a formulation which is more easily described and generalized to treat different boundary conditions. Thus we define* the three-dimensional vectors $\delta_j^{(i)}$ and $\tau_j^{(i)}$ and the 3×3 $L_j^{(i)}$ and $R_j^{(i)}$ by

$$\delta_j^{(i)} \equiv \begin{bmatrix} \delta f_j^{(i)} \\ \delta u_j^{(i)} \\ \delta v_j^{(i)} \end{bmatrix}, \quad \tau_j^{(i)} \equiv \begin{bmatrix} t_{j-1/2}^{(i)} \\ r_{j-1/2}^{(i)} \\ s_{j-1/2}^{(i)} \end{bmatrix} \tag{26a}$$

$$R_j^{(i)} \equiv \begin{bmatrix} \theta_j^{(i)} & \varphi_j^{(i)} & \psi_j^{(i)} \\ 1 & \dfrac{-h_j}{2} & 0 \\ 0 & 1 & \dfrac{-h_j}{2} \end{bmatrix} \quad L_j^{(i)} \equiv \begin{bmatrix} \bar{\theta}_j^{(i)} & \bar{\varphi}_j^{(i)} & \bar{\psi}_j^{(i)} \\ 1 & \dfrac{+h_j}{2} & 0 \\ 0 & 1 & \dfrac{+h_j}{2} \end{bmatrix} \tag{26b}$$

where

$$\begin{cases} \theta_j^{(i)} \equiv \dfrac{h_j}{2b_{j-1/2}^{(i)}}\left[(1+\alpha_n)v_{j-1/2}^{(i)} + \alpha_n v_{j-1/2}^{n-1}\right] & \bar{\theta}_j^{(i)} \equiv -\theta_j^{(i)} \\ \varphi_j^{(i)} \equiv -\dfrac{h_j}{b_{j-1/2}^{(i)}}(\beta+\alpha_n)u_{j-1/2}^{(i)} & \bar{\psi}_j^{(i)} \equiv -\varphi_j^{(i)} \\ \psi_j^{(i)} \equiv 1 + \dfrac{h_j}{2b_{j-1/2}^{(i)}}\left[\dfrac{b_j^{(i)} - b_{j-1}^{(i)}}{h_j} + (1+\alpha_n)f_{j-1/2}^{(i)} - \alpha_n f_{j-1/2}^{n-1}\right] & \\ & \bar{\varphi}^{(i)} \equiv 2 - \psi_j^{(i)} \end{cases} \tag{26c}$$

In terms of these quantities the system (23) is simply

$$R_j^{(i)}\delta_j^{(i)} - L_j^{(i)}\delta_{j-1}^{(\ )} = \tau_j^{(i)} \qquad j = 1, 2, \ldots, J \tag{27}$$

We also write the simple boundary conditions (25) in matrix-vector form to further pave the way for more complicated problems

$$M_0\delta_1^{(i)} = \tau_0^{(i)}, \quad N_J\delta_J^{(i)} = \tau_J^{(i)} \tag{28}$$

Here the (rectangular) matrices and vectors are

$$M_0 \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_0^{(i)} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$N_J \equiv (1 \quad 0 \quad 0), \quad \tau_J^{(i)} \equiv (0) \tag{29}$$

Now the complet linear system (27), (28), which is just (23), (25) in compact form, can be written in compound block-matrix-vector notation as

$$\mathbf{A}^{(i)}\boldsymbol{\delta}^{(i)} = q^{(i)} \tag{30a}$$

where:

$$\mathbf{A}^{(i)} \equiv \begin{bmatrix} M_0 & & & 0 \\ -L_1^{(i)} & R_1^{(i)} & & \\ & \ddots & \ddots & \\ & & -L_J^{(i)} & R_J^{(i)} \\ 0 & & & N_J \end{bmatrix} \quad \boldsymbol{\delta}^{(i)} \equiv \begin{bmatrix} \delta_0^{(i)} \\ \delta_1^{(i)} \\ \vdots \\ \delta_J^{(i)} \end{bmatrix}, \quad \boldsymbol{q}^{(i)} \equiv \begin{bmatrix} \boldsymbol{q}_0^{(i)} \\ \boldsymbol{q}_1^{(i)} \\ \vdots \\ \boldsymbol{q}_J^{(i)} \end{bmatrix} \tag{30b}$$

It is now clear that the nonzero elements in the coefficient matrix above are clustered about the diagonal. Thus various "band matrix" procedures could be invoked to solve the system (30). We show in particular how a block-tridiagonal factorization scheme can be employed, as it is more efficient than general band matric routines in the present case.

The coefficient matrix $\mathbf{A}^{(i)}$ is of order $3J + 3$ and the vector $\boldsymbol{\delta}^{(i)}$ and $\boldsymbol{q}^{(i)}$ have this dimension. We decompose $\boldsymbol{A}^{(i)}$ into 3×3 blocks starting with the upper left hand corner. We also write $\boldsymbol{q}^{(i)}$ in terms of $(J + 1)$ vectors of dimension 3; this is already done for $\boldsymbol{\delta}^{(i)}$. Thus, we rewrite the matrix in Eq. (30b) as

$$\mathbf{A}^{(i)} \equiv \begin{bmatrix} A_0^{(i)} & C_0^{(i)} & & & & & \\ B_1^{(i)} & A_1^{(i)} & C_1^{(i)} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & B_j^{(i)} & A_j^{(i)} & C_j^{(i)} & \\ & & & & & B_J^{(i)} & A_J^{(i)} \end{bmatrix} \tag{31a}$$

where

$$A_0^{(i)} \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\bar{\theta}_j^{(i)} & -\bar{\varphi}_j^{(i)} & -\bar{\psi}_j^{(i)} \end{bmatrix} \quad C_j^{(i)} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \theta_j^{(i)} & \varphi_j^{(i)} & \psi_j^{(i)} \\ 0 \leqq j \leqq J-1 & & \end{bmatrix} \tag{31b}$$

$$A_j^{(i)} \equiv \begin{bmatrix} 1 & \dfrac{-h_j}{2} & 0 \\ 0 & 1 & \dfrac{h_j}{2} \\ -\bar{\theta}_j^{(i)} & -\bar{\varphi}_j^{(i)} & -\psi^{(i)} \end{bmatrix}, \quad 1 \leqq j \leqq J-1 \tag{31c}$$

$$A_J^{(i)} \equiv \begin{bmatrix} 1 & \dfrac{-h_j}{2} & 0 \\ 0 & 1 & \dfrac{-h_j}{2} \\ 0 & 1 & 0 \end{bmatrix} \quad B_j^{(i)} \equiv \begin{bmatrix} -1 & \dfrac{-h_j}{2} & 0 \\ 0 & -1 & \dfrac{-h_j}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad 1 \leqq j \leqq J \tag{31d}$$

$$\boldsymbol{q}_0^{(i)} \equiv \begin{Bmatrix} 0 \\ 0 \\ \begin{bmatrix} t_{1/2}^{(i)} \\ b_{1/2}^{(i)} \end{bmatrix} \end{Bmatrix}, \quad \boldsymbol{q}_j^{(i)} \equiv \begin{Bmatrix} r_{j-1}^{(i)} \\ s_{j-1/2}^{(i)} \\ \begin{bmatrix} t_{j+1/2}^{(i)} \\ b_{j+1/2}^{(i)} \end{bmatrix} \end{Bmatrix}, \quad 1 \leqq j \leqq J-1$$

$$\boldsymbol{q}_J^{(i)} \equiv \begin{bmatrix} r_{J-1/2}^{(i)} \\ s_{J-1/2}^{(i)} \\ 0 \end{bmatrix} \tag{31e}$$

The system (30a) using the structure (31a) can be solved by a more or less standard block tridiagional factorization procedure (see for example Ref. 12, pp. 58–61). For completeness we include here the relevant recursions. First 3×3 matrices $D_j^{(i)}$ and $E_j^{(i)}$ are determined from

$$D_0^{(i)} = A_0^{(i)}; \quad E_j^{(i)} = (D_j^{(i)})^{-1}C_j^{(i)} \quad 0 \leqq j \leqq J-1$$
$$D_j^{(i)} = A_j^{(i)} - B_j^{(i)}E_{j-1}^{(i)}, \quad 1 \leqq j \leqq J \tag{32a}$$

As these matrices are computed the matrices $A_j^{(i)}$ and $C_j^{(i)}$ can be eliminated from the computer storage as they will no longer be required. Next the intermediate 3-vectors

$$\mathbf{Z}_j^{(i)} \equiv [Z_{j,1}^{(i)}, Z_{j,2}^{(i)}, Z_{j,3}^{(i)}]^T$$

are computed from the forward recursion:

$$\mathbf{Z}_0^{(i)} = [D_0^{(i)}]^{-1}\mathbf{q}_0^{(i)}; \quad \mathbf{Z}_j^{(i)} = [D_j^{(i)}]^{-1}[\mathbf{q}_j^{(i)} - B_j^{(i)}\mathbf{Z}_{j-1}^{(i)}], \quad 1 \leqslant j \leqslant J \tag{32b}$$

Finally the solution components $\delta_j^{(i)}$ are obtained from the backward recursion:

$$\boldsymbol{\delta}_J^{(i)} = \mathbf{Z}_J^{(i)}; \quad \boldsymbol{\delta}_j^{(i)} = \mathbf{Z}_j^{(i)} - E_j^{(i)}\boldsymbol{\delta}_{j+1}^{(i)} \quad J-1 \geqq j \geqq 0 \tag{32c}$$

The recursions (32a) require $A_0^{(i)}$ to be nonsingular. It is to achieve this that we have reordered the equations in writing the system as Eq. (27). Conditions to insure that all $D_j^{(i)}$ are nonsingular can be given.

To summarize, one step or iteration of Newton's method is carried out as follows: The quantities $T_{j-1/2}^{n-1}$ are computed from Eq. (18) only once for all of the iterations. Using the latest iterate, the quantities $\{r_{j-1/2}^{(i)}, s_{j-1/2}^{(i)}, t_{j-1/2}^{(i)}\}$ are calculated from Eq. (24). At this stage the inhomogeneous terms $q_j^{(i)}$ are essentially known in Eq. (31e). The matrix elements for $\{A_j^{(i)}, B_j^{(i)}, C_j^{(i)}\}$ are next determined from Eq. (31b–d). The factorization procedure (32a) can now be carried out and then $\{f_j^{(i+1)}, u_j^{(i+1)}, v_j^{(i+1)}\}$ is formed as in Eq. (22).

* In deriving this system we have written Eq. (23c) before Eqs. (23a, b) and divided it through by $b_{j+1/2}^{(i)}$. The division is not necessary and is best avoided.

These calculations are repeated till some convergence criterion is satisfied. Here the wall shear-stress parameter $v(0)$ is used in the convergence criterion and the calculations were stopped when:

$$\left|\frac{\delta v_0^{(i)}}{\frac{1}{2}[v_0^{(i-1)}+v_0^{(i)}]}\right| < \varepsilon_1 \tag{33}$$

It should be noted that Newton's method converges quadratically (for most "smooth" problems) if the initial guess is not too far from the solution. This quadratic convergence is dramatically demonstrated by observing $\delta v_0^{(i)}$ in Eq. (33) during actual calculations. As soon as $\delta v_0^{(i)} \sim 10^{-2}$ say we find that $\delta v_0^{(i+1)} \approx 10^{-4}$ and $\delta v_0^{(i+2)} \approx 10^{-8}$. Frequently, insufficient accuracy in the arithmetic operations will mask this error decay. In such cases the Newton iterates should be terminated at an early stage.

High-Order Accuracy: Richardson Extrapolation

We proceed to show how the accuracy of the above scheme can be improved. The purpose of this increased accuracy is not to acquire more significant digits in the solution. Indeed many digit accuracy is frequently nonsense in practical calculations. The object in using higher order accurate schemes is rather to get reasonable (perhaps crude) answers using relatively few net points. With few net points the iterations converge more rapidly and the computations are generally more efficient. The best method, of course, is that which, for a given prescribed accuracy of the numerical solution, requires the least computing. In this sense the present scheme is much better than others with which we are familiar.

The exact numerical solution of the difference Eq. (18) is a second-order accurate approximation. But since the local truncation error of these schemes has an expansion in powers of $(\Delta\eta_j)^2$ and $(\Delta\xi_n)^2$ and since the scheme is stable, we can in fact show that the errors in approximation have asymptotic expansions of the form (see, for similar results, Ref. 2):

$$f(\xi_n,\eta_j)-f_j^n=\sum_{\nu=0}^{m}\sum_{\mu=0}^{\nu}(h)^{2\mu}(k)^{2\nu-2\mu}e_{\nu\mu}(\xi_n,\eta_j) + 0[(h)^{2m+2}+\ldots+(k)^{2m+2}] \tag{34}$$

Analogous expansions hold for the errors in $u(\xi,\eta)$ and $v(\xi,\eta)$. Here $h \equiv \max h_j$, $k = \max k_n$ and the functions $e_{\nu\mu}(\xi,\eta)$ depend upon the net only through two functions $\theta(\xi)$ and $\varphi(\eta)$ which are such that

$$h_j = \phi(\eta_{j-1/2})h, \quad k_n = \theta(\xi_{n-1/2})k \tag{35}$$

On all nets for which (35) is valid with the same functions $\theta(\xi)$, $\phi(\eta)$ the expansions (35) hold with possibly different h and k values. Examples of such nets are, starting with an arbitrary initial net as in Eq. (4) and then a) dividing each ξ- and η- interval into $r+1$ equal subintervals for some $r = 1, 2, \ldots$; b) dividing *only* the ξ- intervals into $r+1$ equal subintervals; c) reversing ξ and η in case b.

Denote the numerical solution by $\{f_j^n(r), u_j^n(r), v_j^n(r)\}$ for a net with maximal spacings $h^{(r)}$ and $k^{(r)}$. Now let two independent computations be made on nets related as in c above with spacings $(h,k)=(h^{(0)},k)$ and $(h,k)=[h^{(1)},k]$. Then form

$$f_{j_0}^n[h^{(0)},h^{(1)};k] \equiv \frac{h^{(0)}\,f_{j_1}^n[h^{(1)},k]-h^{(1)^2}f_{j_0}^n[h^{(0)},k]}{h^{(0)}-h^{(1)^2}}, \text{ etc.} \tag{36a}$$

and get from Eq. (34)

$$f(\xi_n,\eta^*)-f_{j_0}^n[h^{(0)},h^{(1)};k]=0(k^2+h^4);\ \text{etc.}$$

where $\eta_{j_0}=\eta^*$ on the first net and $\eta_{j_1}=\eta^*$ on the second net. Similar results also hold for u_j^n and v_j^n. Notice that all of the quadratic error terms in $h=\Delta\eta$ are eliminated in this way (i.e., including terms of the form h^2k^2). Of course the combinations (36) can be employed at all points η at which the two nets have common points. Even more accurate approximation can be obtained, regarding the order of the errors in h, if calculations on a third admissible net with $(h,k)\equiv[h^{(2)},k]$ and $\eta_{j_2}=\eta^*$ are made. We now also compute $f_{j_1}^n[h^{(1)},h^{(2)},k]$, etc. in the obvious analogy with Eq. (36a) and then form:

$$f_{j_0}^n[h^{(0)},h^{(1)},h^{(2)};k] \equiv \frac{h^{(0)^2}f_{j_1}[h^{(1)},h^{(2)};k]-h^{(2)^2}f_{j_0}^n[h^{(0)},h^{(1)};k]}{h^{(0)^2}-h^{(2)^2}} \tag{36b}$$

Using Eq. (34) it follows that all terms in h^4 have been eliminated from the error and so

$$f(\xi_n,\eta^*)-f_{j_0}^n[h^{(0)},h^{(1)},h^{(2)};k]\equiv 0(k^2+h^6),\ \text{etc.}$$

Obviously this procedure could be continued, refining the net only in the η-variable. But of course ultimately the errors due to the ξ-variable truncation errors, which are $0(k^2)$ would dominate.

Clearly, the above procedure could be reversed and only the ξ-spacing be defined in an admissible way. More important, this extrapolation can be used to get, say $0(h^6+k^6)$ accuracy if subdivisions of form a are employed on three nets as then

$$\frac{h^{(1)^2}}{h^{(0)^2}-h^{(1)^2}} \quad \frac{k^{(1)^2}}{k^{(0)^2}-k^{(1)^2}}, \text{ etc.}$$

Even higher accuracy can be obtained by additional computations and extrapolations.

The above error reductions assume that exact solutions of the difference equations are employed. Of course this is never the case in practice due to roundoff errors and the fact that the iteration procedure is terminated at a finite stage. Thus, when such inexact numerical solutions are used in the extrapolation a point is rapidly reached beyond which no further improvement is obtained. To get better accuracy either the roundoff or iteration errors (or both) must be reduced. One of the virtues of Newton's method is that with quadratic convergence the iteration error is easily reduced to the same order of magnitude as the roundoff error. Thus, full advantage of extrapolation is more readily obtained than would be the case with first-order iteration schemes.

Very frequently the roundoff error is much less than the truncation errors which are tolerated in the approximate solution. In such circumstances, the iteration error need only be reduced to the same order as the desired truncation error. If the initial error in the iteration scheme (i.e., in the initial guess) is such that $|f_j-f_j^{(0)}|=0(h)$, etc., then by the quadratic convergence of Newton's method it follows that $|f_j-f_j^{(i)}|=0h^2)$. Thus, if the tolerated errors are to be $0(h^{2m})$, it requires only $i=1+(ln\eta/ln2)$ iterations to be consistent. In practice, we seldom require more than two or three iterations (i.e., $m=3$ or 4 corresponding to two or three extrapolations).

Starting Conditions and Estimation of Boundary-Layer Thickness

Starting Conditions

In the present method, the solution begins at the leading edge or at the stagnation point where $\xi=0$. At this point, the flow is laminar and Eq. (11) reduces to the well-known Falkner-Skan equation

$$f'''+ff''+\beta[1-(f')^2]=0 \tag{37}$$

The solution of this equation, say $f^0(\eta)$, subject to the boundary conditions (12) at $\xi=0$ furnishes "initial" conditions $f(0,\eta)=f^0(\eta)$ for solving Eq. (11), Eq. (12) in $\xi>0$, $0<\eta<\eta_\infty$. Many schemes have been devised for solving Eq. (37) [see for example (13)] and in fact, a special case of the present procedure does this very efficiently.

The calculations, either laminar or turbulent, can be started at any fixed downstream ξ-location, say $\xi=\xi_N$, by specifying the initial distributions f_j^N, u_j^N, v_j^N for $0\leqslant j\leqslant J$. In fact, we need only specify the velocity u_j^N for $0\leqslant j\leqslant J$ and the wall shear-stress parameter v_0^N. Then f_j^N and v_j^N are calculated by integration and differentiation from Eq. (14).

Estimation of Boundary-Layer Thickness

Theoretically, the thickness (or edge) of the boundary-layer, $\eta_\infty = \eta_\infty(\xi)$, is the least value of η such that $u(\xi,\eta) = 1$ and $v(\xi,\eta) = 0$ for all $\eta > \eta_\infty(\xi)$. For laminar boundary-layer flows, the thickness is almost constant in the transformed variables; indeed this is one of the reasons for introducing the stretched coordinates. However, for turbulent flows this is not the case and $\eta_\infty(\xi)$ generally increases with ξ. The calculations determine an estimate for the thickness as follows.

It is required that $\eta_\infty(\xi_n) \geqslant \eta_\infty(\xi_{n-1})$ and in fact the calculations start with $\eta_\infty(\xi_0) = \eta_\infty(\xi_1)$ given. When the computations on $\xi = \xi_n$ (for any $n \geqslant 1$) have been completed the test: $|v_J^n| \leqslant \varepsilon_v$ is made where $\eta_J = \eta_\infty(\xi_n)$ and say $\varepsilon_v = 10^{-4}$. If this test is satisfied then set $\eta_\infty(\xi_{n+1}) = \eta_\infty(\xi_n)$, otherwise, set $\eta_\infty(\xi_{n+1}) = \eta_\infty(\xi_n) + \Delta\eta_\infty$ where $\Delta\eta_\infty$ is some constant, say $\Delta\eta_\infty = 0.1 \times \eta_\infty(\xi_0)$. In the latter case values of (f_j^n, f_j^n, v_j^n) for each η_j in $\eta_\infty(\xi_n) < \eta_j \leqslant \eta_\infty(\xi_{n+1})$ are specified by the values $u_j^n \equiv 1$, $v_j^n \equiv 0$ and $f_j^n = \eta_j - \eta_\infty(\xi_n) + f_J^n$, *so that* Eqs. (14a+b) are essentially satisfied.

In chemically reacting flows, the boundary-layer thickness may be much larger and vary somewhat sharper than in the present case. However, a modification of the above procedure, say employing continuous extrapolations of $\eta_\infty(\xi)$, is easily adopted.

Results for Turbulent Flow over a Flat Plate

We present results of some computations with the present method and compare them with results obtained using the method in Ref. 4. This latter method is one of the most successful far devised for turbulent boundary-layer computations, as is shown by comparisons.[1]

For simplicity, we consider turbulent flow over a flat plate and assume that the transformed boundary-layer thickness $\eta_\infty(\xi)$ is a constant, say $\eta_\infty = 24$. At the leading edge $\xi = \xi_0 = 0$ laminar flow is assumed i.e. $b \equiv 1$ in Eq. (11), and at all downstream netpoints $\xi \geqq \xi_1$ the turbulent from (13) is used in Eq. (11). We take 17 nonuniform intervals in the ξ-direction; specifically the ξ-netpoints are: $\xi_n \times 10^7 = 0, 0.86 \times 10^{-3}, 0.43 \times 10^{-2}, 0.86 \times 10^{-2}, 0.43 \times 10^{-1}$, 0.17, 0.26, 0.34, 0.51, 0.68, 0.86, 1.03, 1.28, 1.50, 1.71, 1.88. The method in Ref. 4 employs η-netpoints, $\eta_j = \eta_{j-1} + h_j$, with step lengths of constant ratio; that is $h_j = Kh_{j-1}$. The initial step h_1 is arbitrary and thus the η-netpoints are of the form

$$\eta_j = h_1(K^j - 1)/(K - 1), \quad j = 0, 1, \ldots J \tag{38}$$

For the first set of comparisons, we use $h_1 = 0.05$ and $K = 1.92$ in Eq. (38) to get $J = 10$ intervals across the boundary layer. The final η-step width in this net is approximately, $h_{10} \doteq 12$. We call this net N_0 and it has $18 \times 11 = 198$ net points on the basic rectangle $\{0 \leqslant \xi \leqslant 1.88,\ 0 \leqslant \eta \leqslant 24\}$. We also use the nets N_1 and N_2 in which, respectively, one or two η-points are inserted into each η-interval of the net N_0 (dividing each such interval into 2 or 3 subintervals of equal length). The ξ-spacing is unaltered and thus, N_1 has $18 \times 21 = 378$ netpoints and N_2 has $18 \times 31 = 558$ netpoints on the basic rectangle.

Results of computations on the above nets are given in Tables 1, 2, and 3 where we show, respectively, values of $c_f(\xi_n)$, $R_\theta(\xi_n)$, and $H(\xi_n)$ (see the Nomenclature list for definitions) for the last eight values ξ_n. The extrapolated values using N_0 and N_1 results are in the column headed $N(0,1)$ and the extrapolations using N_0, N_1 and N_2 are in the $N(0,1,2)$ column. The final column, headed N_{23}, are results using the scheme in Ref. 5 on a net with about 230 intervals in the η-direction; thus, N_{23} has approximately $18 \times 231 = 4158$ net points on the basic rectangle. These latter calculations employ double precision arithmetic on an IBM 360/65 and the η_j-netpoints (38) have initial spacing $h_1 = 0.005$ and $K = 1.02022$. In all the calculations, the iteration convergence test (33) used $\varepsilon_1 = 10^{-2}$ and quadratic convergence was observed at all ξ_n points for the present method.

The comparisons in Tables 1, 2 and 3 show that our extrapolated results $N(0,1)$ are for all practical purposes as accurate as those of the method in Ref. 4 applied on the net N_{23}. We recall that $N(0,1)$ required a total of only 576 netpoints while N_{23} has about 4158 netpoints. Also, the calculations with the present method were all single precision. The additional extrapolation to get $N(0,1,2)$ yields somewhat better agreement with the accurate results in N_{23}. However it is likely that the full benefit of the final extrapolation was not obtained due to the very mild convergence criterion on the iterations (i.e., ε_1 should be reduced). Further, we recall that the relatively crude spacing in the ξ-direction eventually masks corrections due solely to refinements in the η-net. Finally, we observe the somewhat remarkable fact that the (unextrapolated) results on the net N_2 (with only 558 netpoints) yield about the same accuracy as the method of Ref. 4 on the net N_{23}.

As a final comparison, we show, in Table 4, the results using the present method and that of Ref. 4 on the same net, N_c, with 27

Table 1 Results for $R_\theta(\xi_n) \times 10^{-3}$ using: present method on three nets N_0, N_1, N_2; two extrapolations $N(0,1)$, $N(0,1,2)$; method of Ref. 4 on the fine net N_{23}

	Present method on nets			Extrapolations		Method of Ref. 4 on
$\xi_n \times 10^{-7}$	N_0	N_1	N_2	$N(0,1)$	$N(0,1,2)$	N_{23}
0.51	0.9922	0.9879	.9832	.9865	.9785	1.0062
0.68	1.2749	1.2347	1.2294	1.2213	1.2257	1.2505
0.86	1.5555	1.4725	1.4056	1.4448	1.4620	1.4840
1.03	1.8330	1.7042	1.6943	1.6613	1.6895	1.7097
1.28	2.2395	2.0428	2.0258	1.9772	2.0166	2.0370
1.50	2.6377	2.3210	2.2944	2.2154	2.2803	2.3017
1.71	2.8987	2.5889	2.5570	2.4856	2.5372	2.5604
1.88	3.1151	2.7985	2.7637	2.6930	2.7410	2.7637

Table 2 Results for $H(\xi_n)$ using: present method on three nets N_0, N_1, N_2; two extrapolations $N(0,1)$, $N(0,1,2)$; method of Ref. 4 on the fine net N_{23}

	Present method on nets			Extrapolations		Method of Ref. 4 on
$\xi_n \times 10^{-7}$	N_0	N_1	N_2	$N(0,1)$	$N(0,1,2)$	N_{23}
0.51	1.5707	1.5085	1.4975	1.4878	1.4888	1.4917
0.68	1.5276	1.4822	1.4705	1.4671	1.4604	1.4655
0.86	1.4983	1.4632	1.4518	1.4515	1.4416	1.4469
1.03	1.4758	1.4488	1.4383	1.4398	1.4287	1.4327
1.28	1.4533	1.4325	1.4232	1.4256	1.4158	1.4170
1.50	1.4248	1.4207	1.4124	1.4193	1.4041	1.4067
1.71	1.4217	1.4110	1.4030	1.4074	1.3953	1.3981
1.88	1.4199	1.4043	1.3966	1.3991	1.3893	1.3922

Table 3 Results for $c_f(\xi_n)10^3$ using: present method on three nets N_0, N_1, N_2; two extrapolations $N(0,1)$, $N(1,1,2)$; method of Ref. 4 on the fine net N_{23}

	Present method on nets			Extrapolations		Method of Ref. 4 on
$\xi_n \times 10^7$	N_0	N_1	N_2	$N(0,1)$	$N(0,1,2)$	N_{23}
0.51	4.3873	4.1221	4.0648	4.0337	4.0171	3.9954
0.68	4.2024	3.9254	3.8708	3.8331	3.8264	3.8002
0.86	4.0585	3.7904	3.7347	3.7010	3.6887	3.6614
1.03	3.9494	3.6828	3.6205	3.5939	3.5678	3.5536
1.28	3.8182	3.5533	3.4850	3.4650	3.4261	3.4244
1.50	3.7479	3.4702	3.4028	3.3776	3.3453	3.3429
1.71	3.7051	3.4003	3.3368	3.2987	3.2844	3.2741
1.88	3.6599	3.3552	3.2873	3.2536	3.2304	3.2272

Table 4 Results for $c_f \times 10^3$ using: present method on net N_c; method of Ref. 5 on net N_c, method of Ref. 4 on N_{23}. Net N_c has $h_1 = 0.05$, $K = 1.19$ and $\eta_\infty = 24$ with $27\eta_f$-points for a total of $18 \times 27 = 486$ net-points

$\xi_n \times 10^7$	Present method on N_c	Method in Ref. 4 on N_c	Method in Ref. 4 on N_{23}
0.51	4.1233	4.7793	3.9954
0.68	3.9370	4.7337	3.8002
0.86	3.8109	4.6592	3.6614
1.03	3.7069	4.5686	3.5536
1.28	3.5818	4.4084	3.4244
1.50	3.5083	4.2792	3.3429
1.71	3.4489	4.1554	3.2741
1.88	3.4037	4.0624	3.2272

points in the η-direction [with $h_1 = 0.05$ and $K = 1.19$ in (38)]. The superiority of the present method is again clearly demonstrated.

Conclusions

Since the writing of this paper, the numerical method described here has been extended to include compressible turbulent boundary layers past two dimensional, axisymmetric bodies and special three dimensional bodies such as infinite swept wings. Comparison of calculated results with those obtained by the method of Ref. 4 shows that the present method provides substantial savings in the computation time. For example, two identical runs for a thick axisymmetric turbulent flow indicates that CPU (Central Processing Unit) time for the present method is $\frac{1}{7}$ the time required by the method of Ref. 4 (on an IBM 360/85).

References

[1] Kline, S. J., Morkovin, M. V., Sovran, G., and Cockrell, D. S., Computation of Turbulent Boundary Layers," AFOSR-IFP-Stanford Conferences, Vol. I and II, 1968, Stanford University Press, Stanford, Calif.

[2] Keller, H. B. and Cebeci, T., "Accurate Numerical Methods for Boundary Layer Flows -I. Two Dimensional Laminar Flows," *Proceedings of Second International Conference on Numerical Methods in Fluid Dynamics*, Springer-Verlag, Berlin, 1971.

[3] Keller, H. B. and Cebeci, T., "An Inverse Problem in Boundary-Layer Flows: Numerical Determination of Pressure Gradient for a Given Wall Shear," *Journal of Computational Physics*, 1972, to be published.

[4] Cebeci, T. and Smith, A. M. O., "A Finite-Difference Method for Calculating Compressible Laminar and Turbulent Boundary Layers," *Journal of Basic Engineering*, Vol. 92, No. 3, Sept. 1970, pp. 523–535.

[5] Keller, H. B., "A New Difference Scheme for Parabolic Problems," *Numerical Solution of Partial Differential Equations*, Vol. II, edited by J. Bramble, Academic Press, New York, 1970.

[6] Van Driest, E. R., "On Turbulent Flow Near a Wall," *Journal of the Aerospace Sciences*, Vol. 23, No. 11, Nov. 1956, pp. 1007–1011.

[7] Cebeci, T., "The Behavior of Turbulent Flow Near a Porous Wall with Pressure Gradient," *AIAA Journal*, Vol. 8, No. 12, Dec. 1970, pp. 2152–2156.

[8] Klebanoff, P. S., "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient," TN 3178, July 1954, NACA.

[9] Cebeci, T., "Calculation of Compressible Turbulent Boundary Layers with Heat and Mass Transfer," *AIAA Journal*, Vol. 9, No. 6, June 1971, pp. 1091–1098.

[10] Hayes, W. D. and Probstein, R. F., *Hypersonic Flow Theory*, Academic Press, New York, 1959, p. 290.

[11] Schlichting, H., *Boundary-Layer Theory*, McGraw-Hill, New York, 1960.

[12] Isaacson, E. and Keller, H. B., *Analysis of Numerical Methods*, Wiley, New York, 1966.

[13] Cebeci, T. and Keller, H. B., "Shooting and Parallel Shooting Methods for Solving the Falkner-Skan Boundary-Layer Equation," *Journal of Computational Physics*, Vol. 7, No. 2, April 1971.